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On the Rates of Approximation of Bernstein Type Operators

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Asymptotic behavior of two Bernstein-type operators is studied in this paper. In the first case, the rate of convergence of a Bernstein operator for a bounded function f is studied at points x where $f(x+)$ and $f(x-)$ exist. In the second case, the rate of convergence of a Szász operator for a function f whose derivative is of bounded variation is studied at points x where $f(x+)$ and $f(x-)$ exist. Estimates of the rate of convergence are obtained for both cases and the estimates are the best possible for continuous points. © 2001 Academic Press

1. INTRODUCTION

For a function f defined on $[0, 1]$ the Bernstein operator B_n is defined by

$$B_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{nk}(x), \qquad p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}. \tag{1}$$

For a function f defined on $[0, \infty)$ the Szász operator S_n is defined by

$$S_n(f, x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) q_{nk}(x), \qquad q_{nk}(x) = e^{-nx} \frac{(nx)^k}{k!}. \tag{2}$$

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In 1983 Cheng [1] proved that

$$B_n(\operatorname{sgn}(t-x), x) = O(n^{-1/6}(x(1-x))^{-5/2}), \qquad x \in (0, 1), \tag{3}$$

where

$$\operatorname{sgn}(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1 & t < 0. \end{cases}$$

This result was later improved by both Guo and Khan [3], and Zeng and Piriou [4].

$$B_n(\operatorname{sgn}(t-x), x) = O(n^{-1/2}(x(1-x))^{-1/2}), \qquad x \in (0, 1). \tag{4}$$

As far as the rate of convergence of Bernstein operator for $|t-x|$ is concerned, Bojanic and Cheng [2] proved the following asymptotic form:

$$B_n(|t-x|, x) = \left(\frac{2x(1-x)}{\pi}\right)^{1/2} \frac{1}{\sqrt{n}} + O(n^{-1}(x(1-x))^{-1/2}), \qquad x \in (0, 1). \tag{5}$$

These equations, Eqs. (3)–(5), have been used to estimate the rate of convergence of operator (1) for functions in $BV[0, 1]$ and functions in $DBV[0, 1] = \{h \mid h' \in BV[0, 1]\}$ (cf. [1–4]). In this paper, using results from probability theory, we shall prove the following result:

$$B_n(\operatorname{sgn}(t-x), x) = \frac{2x - 1 + 6(nx - [nx]) - 3\operatorname{sgn}(nx - [nx])}{3\sqrt{2\pi x(1-x)}\sqrt{n}} + o(n^{-1/2}), \qquad x \in (0, 1). \tag{6}$$

As far as Szász operator is concerned, in 1991 Bojanic and Khan [6] proved that

$$S_n(|t-x|, x) = \left(\frac{2x}{\pi}\right)^{1/2} \frac{1}{\sqrt{n}} + O(n^{-1}). \tag{7}$$

In present paper we shall give a better estimate that

$$n^{3/2}\sqrt{x}\left| S_n(|t-x|, x) - \sqrt{\frac{2x}{\pi}}\frac{1}{\sqrt{n}} \right| \leqslant 2, \qquad x \in [0, \infty). \tag{8}$$

Two classes of functions I_B and I_{DB}, defined as follows, will be considered.

$$I_B = \{f: f \text{ is bounded on } [0, 1]\},$$

and

$$I_{DB} = \Big\{h: h(x) - h(0) = \int_0^x f(t)\, dt,$$
$$f \text{ is bounded in every finite subinterval of } [0, \infty), \quad x \in [0, \infty).\Big\} \tag{9}$$

It is clear that class I_B is more general than $BV[0, 1]$.

We will use the result in Eq. (6) to estimate the rate of convergence of Bernstein operator for $f \in I_B$ at those points x that $f(x+)$ and $f(x-)$ exist. The result in Eq. (8) will be used to estimate the rate of convergence of Szász operator for $h \in I_{DB}$ at those points x that $f(x+)$ and $f(x-)$ exist.

2. RATE OF CONVERGENCE OF BERNSTEIN OPERATORS

In this section we consider the rate of convergence of Bernstein operator (1) for function $f \in I_B$. We introduce the following three quantities first

$$\Omega_{x-}(f, \delta_1) = \sup_{t \in [x - \delta_1, x]} |f(t) - f(x)|,$$

$$\Omega_{x+}(f, \delta_2) = \sup_{t \in [x, x + \delta_2]} |f(t) - f(x)|,$$

$$\Omega(x, f, \lambda) = \sup_{t \in [x - x/\lambda, x + (1 - x)/\lambda]} |f(t) - f(x)|,$$

where $f \in I_B$, $x \in [0, 1]$ is fixed, $0 \leqslant \delta_1 \leqslant x$, $0 \leqslant \delta_2 \leqslant 1 - x$ and $\lambda \geqslant 1$.

It is clear that

(i) $\Omega_{x-}(f, \delta_1)$ and $\Omega_{x+}(f, \delta_2)$ are monotone non-decreasing with respect to δ_1 and δ_2, respectively; $\Omega(x, f, \lambda)$ is monotone non-increasing with respect to λ.

(ii) $\lim_{\delta_1 \to 0+} \Omega_{x-}(f, \delta_1) = 0$, $\lim_{\delta_2 \to 0+} \Omega_{x+}(f, \delta_2) = 0$, $\lim_{\lambda \to \infty} \Omega(x, f, \lambda) = 0$, if f is continuous on the left, continuous on the right, or continuous at the point x, respectively.

(iii) $\Omega_{x-}(f, \delta_1) \leqslant \Omega(x, f, x/\delta_1)$ and $\Omega_{x+}(f, \delta_2) \leqslant \Omega(x, f, (1 - x)/\delta_2)$.

(iv) $\Omega_{x-}(f, \delta_1) \leqslant V_{x-\delta_1}^{x}(f)$, $\Omega_{x+}(f, \delta_2) \leqslant V_{x}^{x+\delta_2}(f)$, $\Omega(x, f, \lambda) \leqslant V_{x-x/\lambda}^{x+(1+x)/\lambda}(f)$, where $V_a^b(f)$ denotes the total variation of the function f which is of bounded variation on $[a, b]$.

The main result of this section is shown below.

THEOREM 1. *Given $f \in I_B$, $f(x+)$ and $f(x-)$ exist at a fixed point $x \in (0, 1)$. Define $g_x(t)$ as*

$$g_x(t) = \begin{cases} f(t) - f(x+), & x < t \leqslant 1; \\ 0, & t = x; \\ f(t) - f(x-), & 0 \leqslant t < x. \end{cases} \tag{10}$$

Then for n sufficiently large we have

$$\left| B_n(f, x) - \frac{f(x+) + f(x-)}{2} - \frac{\mu(f, n, x)}{\sqrt{2\pi\, x(1-x)}\, \sqrt{n}} \right| \leqslant \frac{2}{nx(1-x)} \sum_{k=1}^{n} \Omega(x, g_x, \sqrt{k}) + o(n^{-1/2}), \tag{11}$$

where

$$\mu(f, n, x) = (f(x+) - f(x-))(nx - [nx] + (x-2)/3) + (f(x) - f(x-))(1 - \operatorname{sgn}(nx - [nx])). \tag{12}$$

From Theorem 1 we get immediately

COROLLARY. *Let $f \in I_B$, $f(x+)$ and $f(x-)$ exist at a fixed point $x \in (0, 1)$. If $\Omega(x, g_x, \lambda) = o(\lambda^{-1})$, then*

$$B_n(f, x) = \frac{f(x+) + f(x-)}{2} + \frac{\mu(f, n, x)}{\sqrt{2\pi x(1-x)}\, \sqrt{n}} + o(n^{-1/2}).$$

To prove Theorem 1 we need a few preliminary results.

LEMMA 1. *For $x \in (0, 1)$ there holds*

$$p_{n, [nx]}(x) = \frac{1}{\sqrt{2\pi x(1-x)}\, \sqrt{n}} + o(n^{-1/2}), \tag{13}$$

and

$$p_{n, [nx]+1}(x) = \frac{1}{\sqrt{2\pi x(1-x)}\, \sqrt{n}} + o(n^{-1/2}). \tag{14}$$

Proof. Using Stirling's formula $n! = (n/e)^n \sqrt{2\pi n}\, e^{\theta_n/12n}$, $0 < \theta_n < 1$, we have

$$p_{n,[nx]}(x) - \frac{1}{\sqrt{2\pi x(1-x)}\sqrt{n}}$$
$$= \frac{n!}{[nx]!\,(n-[nx])!} x^{[nx]}(1-x)^{n-[nx]} - \frac{1}{\sqrt{2\pi x(1-x)}\sqrt{n}}$$
$$= \frac{1}{\sqrt{2\pi x(1-x)}\sqrt{n}} \left(e^{c(n,x)} \left(\frac{nx}{[nx]}\right)^{[nx]+1/2} \left(\frac{n-nx}{n-[nx]}\right)^{n-[nx]+1/2} - 1 \right),$$

where $e^{c(n,x)} = e^{\theta_n/12n - \theta_{[nx]}/12[nx] - \theta_{n-[nx]}/12(n-[nx])} \to 1$ $(n \to +\infty)$.

Straightforward computation shows that

$$\lim_{n\to\infty} \left(\frac{nx}{[nx]}\right)^{[nx]+1/2} \left(\frac{n-nx}{n-[nx]}\right)^{n-[nx]+1/2} = 1.$$

Hence, (13) holds. (14) can be proved similarly.

The following Lemma is an asymptotic form of the central limit theorem in probability theory. Its proof and further discussion can be found in Feller [5, pp. 540–542].

LEMMA 2. *Let* $\{\xi_k\}_{k=1}^{\infty}$ *be a sequence of independent and identically distributed random variables with the expectation* $E(\xi_1) = a_1$, *the variance* $E(\xi_1 - a_1)^2 = \sigma^2 > 0$, $E(\xi_1 - a_1)^3 < \infty$, *and let* F_n *stand for the distribution function of* $\sum_{i=1}^{n} (\xi_i - a_1)/\sigma\sqrt{n}$. *If* F_n *is a lattice distribution and* $F_n^{\#}$ *is a polygonal approximant of* F_n (*see Definition* 1), *then the following equation holds for all the points of* F_n

$$F_n^{\#}(t) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\mu^2/2}\, du - \frac{E(\xi_1 - a_1)^3}{6\sigma^3\sqrt{n}} (1-t^2) \frac{1}{\sqrt{2\pi}} e^{-t^2/2} = o(n^{-1/2}). \tag{15}$$

DEFINITION 1 [5, p. 540, Definition]. Let F be concentrated on the lattice of points $b \pm nh$, but on no sublattice (that is, h is the span of F). A polygonal approximant $F^{\#}$ to F is a distribution function with a polygonal graph with vertices at the midpoints $b \pm (n + 1/2)h$ lying on the graph of F. Thus

$$F^{\#}(t) = F(t) \qquad \text{if} \quad t = b \pm (n+1/2)h; \tag{16}$$

$$F^{\#}(t) = \tfrac{1}{2}[F(t) + F(t-)] \qquad \text{if} \quad t = b \pm nh. \tag{17}$$

LEMMA 3. *For every $x \in (0, 1)$ the following equation holds*

$$B_n(\operatorname{sgn}(t-x), x) = \frac{2x-1+6(nx-[nx])-3\operatorname{sgn}(nx-[nx])}{3\sqrt{2\pi x(1-x)}\sqrt{n}} + o(n^{-1/2}). \tag{18}$$

Proof. Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of independent random variables with the same two-point distribution $P(\xi_i = j) = x^j(1-x)^{1-j}$, $j = 0, 1$ and $x \in [0, 1]$ is a parameter. Direct calculation gives $E\xi_1 = x$, $E(\xi_1 - E\xi_1)^2 = x(1-x)$ and $E(\xi_1 - E\xi_1)^3 = x(1-x)(1-2x) < \infty$ (cf. [7, p. 14]). Let $\eta_n = \sum_{i=1}^{n} \xi_i$ and F_n stands for the distribution function *of* $\sum_{i=1}^{n}(\xi_i - a_1)/\sigma\sqrt{n}$. Then the probability distribution of the random variable η_n is

$$P(\eta_n = k) = \binom{n}{k} x^k(1-x)^{n-k} = p_{nk}(x).$$

Hence

$$\begin{aligned} B_n(\operatorname{sgn}(t-x), x) &= -\sum_{k<nx} p_{nk}(x) + \sum_{k>nx} p_{nk}(x) \\ &= -\sum_{k<nx} p_{nk}(x) - \sum_{k\leqslant nx} p_{nk}(x) + 1 \\ &= -P(\eta_n < nx) - P(\eta_n \leqslant nx) + 1 = -F_n(0-) - F_n(0) + 1 \\ &= 2F_n^{\#}(0) - F_n(0-) - F_n(0) + 1 - 2F_n^{\#}(0). \end{aligned} \tag{19}$$

From Lemma 2 we get

$$\begin{aligned} 1 - 2F_n^{\#}(0) &= -\frac{2E(\xi_1 - a_1)^3}{6\sigma^3\sqrt{n}} \frac{1}{\sqrt{2\pi}} + o(n^{-1/2}) \\ &= \frac{2x-1}{3\sqrt{2\pi x(1-x)}} \frac{1}{\sqrt{n}} + o(n^{-1/2}). \end{aligned} \tag{20}$$

In the following we estimate $2F_n^{\#}(0) - F_n(0-) - F_n(0)$.

If $nx = [nx]$, then 0 is a lattice point of F. From (17) we get

$$2F_n^{\#}(0) - F_n(0-) - F_n(0) = 0.$$

If $nx \neq [nx]$, then

$$F_n(0) = F_n(0-) = \sum_{k \leqslant [nx]} p_{nk}(x),$$

and we know distribution function F is a stepfunction. Hence $F_n(t) = \sum_{k \leqslant [nx]} p_{nk}(x)$ on the interval $[([nx]-nx)/\sigma\sqrt{n}, ([nx]+1-nx)/\sigma\sqrt{n})$.

For $0 < nx - [nx] \leqslant 1/2$, from (17) and (16) it is known that

$$F_n^{\#}\left(\frac{[nx]-nx}{\sigma\sqrt{n}}\right) = \frac{1}{2}\left(\sum_{k \leqslant [nx]-1} p_{nk}(x) + \sum_{k \leqslant [nx]} p_{nk}(x)\right),$$

and

$$F_n^{\#}\left(\frac{[nx]-nx+1/2}{\sigma\sqrt{n}}\right) = \sum_{k \leqslant [nx]} p_{nk}(x).$$

By direct calculation we get the expression of $F_n^{\#}(t)$ on interval $\lfloor([nx]-nx)/\sigma\sqrt{n}, ([nx]-nx+1/2)/\sigma\sqrt{n}\rfloor$

$$F_n^{\#}(t) = \sigma\sqrt{n}\, p_{n,[nx]}(x)\, t + \sum_{k \leqslant [nx]} p_{nk}(x) + (nx - [nx] - 1/2)\, p_{n,[nx]}(x).$$

Hence, for $0 < nx - [nx] \leqslant 1/2$

$$F_n^{\#}(0) = \sum_{k \leqslant [nx]} p_{nk}(x) + (nx - [nx] - 1/2)\, p_{n,[nx]}(x).$$

Similarly, for $1/2 < nx - [nx] < 1$

$$F_n^{\#}(0) = \sum_{k \leqslant [nx]} p_{nk}(x) + (nx - [nx] - 1/2)\, p_{n,[nx]+1}(x).$$

Consequently

$$2F_n^{\#}(0) - F_n(0-) - F_n(0)$$
$$= \begin{cases} 0, & nx = [nx] \\ (2nx - 2[nx] - 1)\, p_{n,[nx]}(x), & [nx] < nx \leqslant [nx] + 1/2 \\ (2nx - 2[nx] - 1)\, p_{n,[nx]+1}(x), & [nx] + 1/2 < nx < [nx] + 1. \end{cases} \tag{21}$$

Now (18) follows by combining (19)–(21) with Lemma 1.

Proof of Theorem 1. For any $f \in I_B$, if $f(x+)$ and $f(x-)$ exist at x, we decompose f into

$$f(t) = \frac{f(x+)+f(x-)}{2} + g_x(t) + \frac{f(x+)-f(x-)}{2}\operatorname{sgn}(t-x)$$
$$+ \delta_x(t)\left[f(x) - \frac{f(x+)+f(x-)}{2}\right], \tag{22}$$

where $g_x(t)$ is defined in (10) and

$$\delta_x(t)=\begin{cases}1, & t=x\\ 0, & t\neq x.\end{cases}$$

Direct calculation gives

$$B_n(\delta_x, x)=(1-\operatorname{sgn}(nx-[nx]))\; p_{n,[nx]}(x). \tag{23}$$

From (22), (23), Lemmas 1, 2, and simple computation we get

$$\left|B_n(f, x)-\frac{f(x+)+f(x-)}{2}-\frac{\mu(f, n, x)}{\sqrt{2\pi\, x(1-x)}}\right|\leqslant |B_n(g_x, x)|+o(n^{-1/2}), \tag{24}$$

where μ is defined in (12).

Next we estimate $|B_n(g_x, x)|$. Recall the Lebesgue–Stieltjes integral representation. We have

$$B_n(g_x, x)=\int_0^1 g_x(t)\, d_tK_n(x, t), \tag{25}$$

where

$$K_n(x, t)=\begin{cases}\sum_{k\leqslant nt} p_{nk}(x), & 0<t\leqslant 1\\ 0, & t=0.\end{cases}$$

We decompose the integral of (25) into three parts as follows:

$$\int_0^1 g_x(x)\, d_tK_n(x, t)=\Delta_{1,n}(g_x)+\Delta_{2,n}(g_x)+\Delta_{3,n}(g_x),$$

where

$$\Delta_{1,n}(g_x)=\int_0^{x-x/\sqrt{n}} g_x(t)\, d_tK_n(x, t),$$

$$\Delta_{2,n}(g_x)=\int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} g_x(t)\, d_tK_n(x, t)$$

$$\Delta_{3,n}(g_x)=\int_{x+(1-x)/\sqrt{n}}^1 g_x(t)\, d_tK_n(x, t).$$

We shall evaluate $\Delta_{1,n}(g_x)$, $\Delta_{2,n}(g_x)$ and $\Delta_{3,n}(g_x)$ with the quantities $\Omega_{x-}(g_x, \delta_1)$, $\Omega_{x+}(g_x, \delta_2)$ and $\Omega(x, g_x, \lambda)$ (for simplicity, in the following we shall write them as $\Omega_{x-}(\delta_1)$, $\Omega_{x+}(\delta_2)$ and $\Omega(x, \lambda)$, respectively). First, for $\Delta_{2,n}(g_x)$ note that $g_x(x)=0$, we have

$$|\Delta_{2,n}(g_x)| \leqslant \int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} |g_x(t)-g_x(x)|\, d_tK_n(x,t) \leqslant \Omega(x, \sqrt{n}). \tag{26}$$

To estimate $|\Delta_{1,n}(g_x)|$, note that $\Omega_{x-}(\delta_1)$ is monotone non-decreasing for δ_1, hence it follows that

$$|\Delta_{1,n}(g_x)| = \left| \int_0^{x-x/\sqrt{n}} g_x(t)\, d_tK_n(x,t) \right| \leqslant \int_0^{x-x/\sqrt{n}} \Omega_{x-}(x-t)\, d_tK_n(x,t).$$

Using partial integration with $y = x - x/\sqrt{n}$, we have

$$\begin{aligned} &\int_0^{x-x/\sqrt{n}} \Omega_{x-}(x-t)\, d_tK_n(x,t) \\ &\qquad \leqslant \Omega_{x-}(x-y)\, K_n(x, y+) + \int_0^y \hat{K}_n(x,t)\, d(-\Omega_{x-}(x-t)), \end{aligned} \tag{27}$$

where $\hat{K}_n(x,t)$ is the normalized form of $K_n(x,t)$. Since $\hat{K}_n(x,t) \leqslant K_n(x,t)$ and $K_n(x, y+) = K_n(x,y)$ on $(0,1)$, from (27) and using the well-known result $\hat{K}_n(x,t) \leqslant K_n(x,t) \leqslant \sum_{k \leqslant nt} p_{nk}(x) \leqslant x(1-x)/n(t-x)^2$ it follows that

$$|\Delta_{1,n}(g_x)| \leqslant \Omega_{x-}(x-y)\, \frac{x(1-x)}{n(x-y)^2} + \frac{x(1-x)}{n} \int_0^y \frac{1}{(x-t)^2}\, d(-\Omega_{x-}(x-t)). \tag{28}$$

With the fact that

$$\begin{aligned} &\int_0^y \frac{1}{(x-t)^2}\, d(-\Omega_{x-}(x-t)) \\ &\quad = -\frac{1}{(x-t)^2}\, \Omega_{x-}(x-t) \Big|_0^y + \int_0^y \Omega_{x-}(x-t)\, \frac{2}{(x-t)^3}\, dt \\ &\quad = -\frac{1}{(x-y)^2}\, \Omega_{x-}(x-y) + \frac{1}{x^2}\, \Omega_{x-}(x) + \int_0^y \Omega_{x-}(x-t)\, \frac{2}{(x-t)^3}\, dt, \end{aligned}$$

we have from (28)

$$|\Delta_{1,n}(g_x)| \leqslant \frac{x(1-x)}{nx^2}\, \Omega_{x-}(x) + \frac{x(1-x)}{n} \int_0^{x-x/\sqrt{n}} \Omega_{x-}(x-t)\, \frac{2}{(x-t)^3}\, dt.$$

Putting $t = x - x/\sqrt{u}$ for the last integral we get

$$\int_0^{x - x/\sqrt{n}} \Omega_{x-}(x-t) \frac{2}{(x-t)^3} \, dt = \frac{1}{x^2} \int_1^n \Omega_{x-}(x/\sqrt{u}) \, du.$$

Consequently

$$|\varDelta_{1,n}(g_x)| \leqslant \frac{1-x}{nx} \left(\Omega_{x-}(x) + \int_1^n \Omega_{x-}(x/\sqrt{u}) \, du \right). \tag{29}$$

Using a similar method to estimate $|\varDelta_{3,n}(g_x)|$, we get

$$|\varDelta_{3,n}(g_x)| \leqslant \frac{x}{n(1-x)} \left(\Omega_{x+}(1-x) + \int_1^n \Omega_{x+}((1-x)/\sqrt{u}) \, du \right). \tag{30}$$

From (26), (29) and (30) it follows that

$$|B_n(g_x, x)| \leqslant \Omega(x, \sqrt{n}) + \left(\frac{1-x}{nx} + \frac{x}{n(1-x)} \right) \left(\Omega(x, 1) + \int_1^n \Omega(x, \sqrt{u}) \, du \right). \tag{31}$$

By monotonicity of $\Omega(x, \lambda)$ and the fact that $(1-x)^2 + x^2 \leqslant 1$, $1/(n-1) \leqslant 1/nx(1-x)$ $(n > 1)$ we have

$$\begin{aligned} |B_n(g_x, x)| &\leqslant \frac{1}{n-1} \sum_{k=2}^n \Omega(x, \sqrt{k}) + \frac{1}{nx(1-x)} \Omega(x, 1) \\ &\quad + \frac{1}{nx(1-x)} \sum_{k=1}^n \Omega(x, \sqrt{k}) \\ &\leqslant \frac{2}{nx(1-x)} \sum_{k=1}^n \Omega(x, \sqrt{k}). \end{aligned} \tag{32}$$

Theorem 1 now follows from (24) and (32).

3. RATE OF CONVERGENCE OF SZÁSZ OPERATORS

In this section we consider the rate of convergence of Szász operator (2) for function $h \in I_{DB}$ (defined in (9)). First we introduce the quantity

$$\Omega^*(x, f, \delta) = \sup_{t \in [x-\delta, x+\delta]} |f(t) - f(x)|,$$

where f is bounded in every finite subinterval of $[0, \infty)$.

The main result of this section is as follows:

THEOREM 2. *Let h be a function in I_{DB} and let $h(t)=O(e^{\alpha t \log t})$ for some $\alpha>0$ as $t\to\infty$. If $f(x+)$ and $f(x-)$ exist at a fixed point $x\in(0,\infty)$, and we write $\tau=f(x+)-f(x-)$, then for n sufficiently large we have*

$$\left|S_n(h,x)-h(x)-\tau(x/2\pi)^{1/2}\frac{1}{\sqrt{n}}\right|\leqslant\frac{|\tau|}{n^{3/2}x^{1/2}}+\frac{4x+2}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega^*(x,\varphi_x,x/k)$$
$$+O(1)\frac{(2x+1)^{(2x+1)\alpha}}{1+\sqrt{nx}}(e/4)^{nx}, \qquad (33)$$

where $[\sqrt{n}]$ is the greatest integer not exceeding $\sqrt{n}$ and $\varphi_x(t)$ is defined as

$$\varphi_x(t)=\begin{cases} f(t)-f(x+), & x<t<\infty;\\ 0, & t=x;\\ f(t)-f(x-), & 0\leqslant t<x.\end{cases}$$

In view of the fact that $(1/\sqrt{n})\sum_{k=1}^{[\sqrt{n}]}\Omega(x,\varphi_x,k)\to 0\ (n\to\infty)$, from Theorem 2 we get the following asymptotic formula

$$S_n(h,x)=h(x)+(x/2\pi)^{1/2}\frac{\tau}{\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right), \qquad (34)$$

if h satisfies the assumptions of Theorem 2. In particular, (34) is true for $h\in DBV[0,\infty)$. For Bernstein operator $B_n(h,x)$ Bojanic and Cheng [2] proved a similar asymptotic formula for $h\in DBV[0,1]$.

The following lemma is needed for proving Theorem 2.

LEMMA 4. *For $x\in[0,\infty)$ there holds*

$$n^{3/2}\sqrt{x}\left|S_n(|t-x|,x)-\sqrt{\frac{2x}{\pi}}\frac{1}{\sqrt{n}}\right|\leqslant 2. \qquad (35)$$

Proof. By the fact that $S_n(t,x)=x$ we have

$$\begin{aligned} S_n(|t-x|,x)&=2\sum_{k=0}^{[nx]}\left(x-\frac{k}{n}\right)\frac{(nx)^k}{k!}e^{-nx}\\ &=2\sum_{k=0}^{[nx]}x\frac{(nx)^k}{k!}e^{-nx}-2\sum_{k=0}^{[nx]-1}x\frac{(nx)^k}{k!}e^{-nx}\\ &=2xe^{-nx}\frac{(nx)^{[nx]}}{[nx]!}.\end{aligned}$$

If $x<1/n$, then $[nx]=0$. Obviously, $0\leqslant 2\sqrt{nx}\,e^{-nx}\leqslant 2$.

Hence

$$n^{3/2}\sqrt{x}\left|S_n(|t-x|,x)-\sqrt{\frac{2x}{\pi}}\frac{1}{\sqrt{n}}\right|=nx\left|2\sqrt{nx}\,e^{-nx}-\sqrt{\frac{2}{\pi}}\right|\leqslant 2.$$

If $x\geqslant 1/n$, then $[nx]\geqslant 1$. Using Stirling's formula $n!=(n/e)^n\sqrt{2\pi n}\,e^{\theta_n/12n}$, $0<\theta_n<1$, we get

$$\begin{aligned}
&n^{3/2}\sqrt{x}\left(S_n(|t-x|,x)-\sqrt{\frac{2x}{\pi}}\frac{1}{\sqrt{n}}\right)\\
&\quad=\sqrt{\frac{2}{\pi}}\,nx\left(e^{-nx+[nx]}\left(\frac{nx}{[nx]}\right)^{[nx]+1/2}e^{c}-1\right)\\
&\quad=\sqrt{\frac{2}{\pi}}\,nx(e^{c}-1)+e^{c}\sqrt{\frac{2}{\pi}}\,nx\left(e^{-nx+[nx]}\left(\frac{nx}{[nx]}\right)^{[nx]+1/2}-1\right),
\end{aligned}$$

where

$$e^{-1/(12[nx])}\leqslant e^{c}\leqslant 1. \tag{36}$$

Thus, from expansion formula $e^{c}=\sum_{i=0}^{\infty}c^{i}/i!$, it is not difficult to show that

$$nx\,|e^{c}-1|\leqslant 1/4.$$

On the other hand, write $nx=[nx]+\varepsilon$ $(0\leqslant\varepsilon<1)$, then

$$\begin{aligned}
&e^{c}\sqrt{\frac{2}{\pi}}\,nx\left|e^{-nx+[nx]}\left(\frac{nx}{[nx]}\right)^{[nx]+1/2}-1\right|\\
&\quad=e^{c}\sqrt{\frac{2}{\pi}}\frac{nx}{[nx]}[nx]\left|e^{-\varepsilon}\left(1+\frac{\varepsilon}{[nx]}\right)^{[nx]+1/2}-1\right|\\
&\quad=e^{c}\sqrt{\frac{2}{\pi}}\frac{nx}{[nx]}[nx]\left(e^{-\varepsilon}\left(1+\frac{\varepsilon}{[nx]}\right)^{[nx]+1/2}-1\right)\\
&\quad\leqslant 2\sqrt{\frac{2}{\pi}}[nx]\left(e^{-\varepsilon}\left(1+\frac{\varepsilon}{[nx]}\right)^{[nx]+1/2}-1\right).
\end{aligned}$$

It is easy to verify that

$$[nx]\left(e^{-\varepsilon}\left(1+\frac{\varepsilon}{[nx]}\right)^{[nx]+1/2}-1\right)\leqslant\varepsilon\leqslant 1.$$

Consequently

$$n^{3/2}\sqrt{x}\left|S_n(|t-x|,x)-\sqrt{\frac{2x}{\pi}}\frac{1}{\sqrt{n}}\right|\leqslant\frac{\sqrt{2}}{4\sqrt{\pi}}+\frac{2\sqrt{2}}{\sqrt{\pi}}\leqslant 2.$$

The proof of Lemma 4 is completed.

Proof of Theorem 2. By straightforward computation we find that (cf. [2, pp. 138–139])

$$S_n(h,x)-h(x)=\frac{f(x+)-f(x-)}{2}S_n(|t-x|,x)$$
$$-L_n(h,x)+R_n(h,x)+Q_n(h,x), \tag{37}$$

where

$$L_n(h,x)=\sum_{k<nx}\left(\int_{k/n}^{x}\varphi_x(t)\,dt\right)q_{nk}(x),$$
$$R_n(h,x)=\sum_{nx<k\leqslant 2nx}\left(\int_{x}^{k/n}\varphi_x(t)\,dt\right)q_{nk}(x)$$

and

$$Q_n(h,x)=\sum_{k>2nx}\left(\int_{k/n}^{x}\varphi_x(t)\,dt\right)q_{nk}(x).$$

Define

$$\tilde{K}_n(x,t)=\sum_{k\leqslant nt}q_{nk}(x),\qquad 0\leqslant t\leqslant x.$$

Then

$$L_n(h,x)=\int_0^x\left(\int_t^x\varphi_x(v)\,dv\right)d_t\tilde{K}_n(x,t)+\left(\int_0^x\varphi_x(v)\,dv\right)\tilde{K}_n(x,0)$$
$$=\left(\int_t^x\varphi_x(v)\,dv\right)\tilde{K}_n(x,t)\Big|_0^x+\int_0^x\tilde{K}_n(x,t)\,\varphi_x(t)\,dt$$
$$+\left(\int_0^x\varphi_x(v)\,dv\right)\tilde{K}_n(x,0)$$
$$=\int_0^x\tilde{K}_n(x,t)\,\varphi_x(t)\,dt=\left(\int_0^{x-x/\sqrt{n}}+\int_{x-x/\sqrt{n}}^{x}\right)\tilde{K}_n(x,t)\,\varphi_x(t)\,dt.$$

Since $\varphi_x(x)=0$, $\tilde{K}_n(x, t)\leqslant 1$, by monotonicity of $\Omega^*(x, \varphi_x, \delta)$ we have

$$\left|\int_{x-x/\sqrt{n}}^{x} \tilde{K}_n(x, t)\, \varphi_x(t)\, dt\right| \leqslant \frac{x}{\sqrt{n}}\, \Omega^*(x, \varphi_x, x/\sqrt{n}) \leqslant \frac{2x}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega^*(x, \varphi_x, x/k).$$

Again, for $t<x$ it is known that $\tilde{K}_n(x, t)=\sum_{k\leqslant nt} q_{nk}(x)\leqslant (1/(x-t)^2)$ $S_n((t-x)^2, x)\leqslant x/n(x-t)^2$. Hence

$$\left|\int_{0}^{x-x/\sqrt{n}} \tilde{K}_n(x, t)\, \varphi_x(t)\, dt\right| \leqslant \frac{x}{n} \int_{0}^{x-x/\sqrt{n}} \Omega^*(x, \varphi_x, x-t)\, \frac{dt}{(x-t)^2}.$$

Replacing the variable t by $x-x/u$ for the last integral, then

$$\left|\int_{0}^{x-x/\sqrt{n}} \tilde{K}_n(x, t)\, \varphi_x(t)\, dt\right| \leqslant \frac{x}{nx} \int_{1}^{\sqrt{n}} \Omega^*(x, \varphi_x, x/u)\, du$$

$$\leqslant \frac{1}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega^*(x, \varphi_x, x/k).$$

Consequently

$$|L_n(h, x)| \leqslant \frac{2x+1}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega^*(x, \varphi_x, x/k). \tag{38}$$

A similar estimate gives

$$|R_n(h, x)| \leqslant \frac{2x+1}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega^*(x, \varphi_x, x/k). \tag{39}$$

Finally by the assumption $h(t)=O(e^{\alpha t \log t})$ *for some* $\alpha>0$ *as* $t\to\infty$, and by direct computation it can be shown that (cf. [9, (31), p. 320])

$$|Q_n(h, x)| = O(1)\, \frac{(2x+1)^{(2x+1)\alpha}}{1+\sqrt{nx}}\, (e/4)^{nx}. \tag{40}$$

Theorem 2 now follows by combining (37)–(40) with Lemma 4.

Remark. If f is continuous at x, then (11) becomes

$$|B_n(f, x)-f(x)| \leqslant \frac{2}{nx(1-x)} \sum_{k=1}^{n} \Omega(x, f, \sqrt{k}), \tag{41}$$

and (33) becomes

$$|S_n(h, x) - h(x)| \leqslant \frac{4x+2}{n} \sum_{k=1}^{[\sqrt{n}]} \Omega^*(x, \varphi_x, x/k) + O(1) \frac{(2x+1)^{(2x+1)\alpha}}{1+\sqrt{nx}} (e/4)^{nx}. \tag{42}$$

Inequalities (41) and (42) are the best possible we can get in the sense that they cannot be improved any further asymptotically (see [1, 4, and 8]).

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